A STUDYON FUNCTIONAL EQUATIONS RELATED TO DERIVATION ON UNITAL SEMIPRIME RINGS

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Abstract:

The aim of this paper is to establish the following result: Consider RRR as an n!-torsion-free semiprime unital ring. Let D,G:R→RD, G: R \to RD,G:R→R be additive mappings that satisfy the conditions D(xn)=D(xn−1)x+xn−1G(x)D(x^n)= D(x^{n-1})x + x^{n-1}G(x)D(xn)=D(xn−1)x+xn−1G(x) G(xn)=G(xn−1)x+xn−1D(x)G(x^n) = G(x^{n-1})x + x^{n-1}D(x)G(xn)=G(xn−1)x+xn−1D(x) for all x∈Rx \in Rx∈R and some integer n>1n > 1n>1. Under these conditions, it is shown that DDD and GGG are derivations and that D=GD = GD=G.

Keywords: Functional Identity, Derivation, Jordan Derivation, Semiprime Ring.

Introduction

Throughout, R will represent an associative ring with center Z(R). As usual we write [x, y] for xy−yx. Given an integer n ≥ 2, a ring R is said to be n-torsion free, if for x ∈ R, nx = 0 implies x = 0. Recall that a ring R is prime, if for a, b ∈ R, aRb = (0) implies a = 0 or b = 0 and is semiprime in case aRa = (0) implies a = 0. We denote by Qs the symmetric Martindale ring of quotients. For explanation of Qs, we refer the reader to [1]. An additive mapping D : R → R is called a derivation if D(xy) = D(x)y +xD(y) holds for all pairs x, y ∈ R and is called a Jordan derivation in case D(x 2) = D(x)x + xD(x) is fulfilled for all x ∈ R. A derivation D is inner in case there exists such a ∈ R that D(x) = [x, a] holds for all x ∈ R. Every derivation is a Jordan derivation, but the converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [2]. Cusack [5] generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). An additive mapping D : R → R, where R is an arbitrary ring, is called a Jordan triple derivation in case D(xyx) = D(x)yx+xD(y)x+xyD(x) holds for all pairs x, y ∈ R. One can easily prove that any Jordan derivation on an arbitrary 2- torsion free ring R is a Jordan triple derivation (see for example [2]) Brešar [4] has proved the following result.

Theorem A

Let R be a 2-torsion-free semiprime ring, and let D:R→RD: R \to RD:R→R be a Jordan triple derivation. Under these conditions, DDD is necessarily a derivation.

As previously noted, every Jordan derivation D on a 2-torsion-free ring is a Jordan triple derivation. Theorem A extends Cusack's version of Herstein's theorem.

Based on Theorem A, Vukman, Kosi-Ulbl, and Eremita [9] demonstrated the following conclusion.

Theorem B. Let R be a 2-torsion free semiprime ring. Suppose there exists an additive mapping T : R → R such that

$$T(xyx) = T(x)yx - xT(y)x + xyT(x)$$

for all x, y ∈ R. Then there exists q ∈ Qs such that 2T(x) = qx + xq, for all x ∈ R

In the same publication, Vukman, Kosi-Ulbl, and Eremita established the following conclusion, which is an immediate consequence of Theorems A and B.

Theorem C. Let R be a 2-torsion free semiprime ring. If D, G : R → R are additive mappings such that

$$D(xyx) = D(x)yx - xG(y)x + xyD(x)$$
$$G(xyx) = G(x)yx - xD(y)x + xyG(x)$$

for all x, y ∈ R. Then there exists a derivation S : R → R and q ∈ Qs such that

$$4D(x) = qx + xq + S(x)$$
$$4G(x) = qx + xq - S(x)$$

for all $x \in R$.

Motivated by Theorem A, Vukman [10] recently proved the following result.

Theorem D. Let R be a 2-torsion free semiprime ring and let D : R → R be an additive mapping. Suppose that either of the relations

$$D(xyx) = D(xy)x + xyD(x), \qquad (1.1)$$
$$D(xyx) = D(x)yx + xD(yx)$$

holds for all pairs x, y ∈ R. In both cases D is a derivation. The substitution y = x n−2 in relations (1.1), gives

$$D(x^n) = D(x^{n-1})x + x^{n-1}D(x), \qquad (1.2)$$
$$D(x^n) = D(x)x^{n-1} + xD(x^{n-1}).$$

This study aims to show a result related to functional equation (1.2) and confirm our hypothesis in [7] for semiprime unital rings.

Theorem 1.1. Let R be n!-torsion free semiprime unital ring and let D, G : R → R be additive mappings satisfying either the relations

$$D(x^n) = D(x^{n-1})x + x^{n-1}G(x),$$
$$G(x^n) = G(x^{n-1})x + x^{n-1}D(x)$$

or the relations

$$D(x^n) = D(x)x^{n-1} + xG(x^{n-1}),$$
$$G(x^n) = G(x)x^{n-1} + xD(x^{n-1})$$

for all $x \in R$ and some integer $n > 1$. In both cases D and G are derivations and D = G.

Proof. We will focus our attention on the first set of relationships. The evidence for the second system of relations is similar and hence will be omitted. We have.

$$D(x^n) = D(x^{n-1})x + x^{n-1}G(x), \qquad (1.3)$$
$$G(x^n) = G(x^{n-1})x + x^{n-1}D(x).$$

Subtracting the two relations of equation (1.3), we obtain

$$T(x^n) = T(x^{n-1})x - x^{n-1}T(x), \qquad (1.4)$$

where T = D − G. We will denote the identity element of the ring R by e. Putting e for x in the above relation gives

$T(e) = 0.$ (1.5)

Let y be any element of the center Z(R). Putting x + y for x in the relation (1.4), we get

$$\sum_{i=0}^{n} \binom{n}{i} T(x^{n-i}y^i) = \left(\sum_{i=0}^{n-1} \binom{n-1}{i} T(x^{n-1-i}y^i)\right)(x+y)$$
$$-\left(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\right) T(x+y).$$

Using (1.4) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where fi(x, y) stands for the expression of terms involving i factors of y, that is

$$\begin{aligned} f_i(x,y) &= \binom{n}{i} T(x^{n-i}y^i) \\ &\quad - \binom{n-1}{i}\Big(T(x^{n-1-i}y^i)x - (x^{n-1-i}y^i)T(x)\Big) \\ &\quad - \binom{n-1}{i-1}\Big(T(x^{n-i}y^i)y - (x^{n-i}y^i)T(y)\Big). \end{aligned}$$

Replacing x by x + 2y, x + 3y,..., x + (n − 1)y in turn in the relation (1.4) and expressing the resulting system of (n − 1) homogeneous equations of variables fi(x, y), i = 1, 2, . . . , n − 1, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the above matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular putting the identity element e for y, we obtain

$$f_{n-1}(x,e) = \binom{n}{n-1}T(x) - \binom{n-1}{n-1}\Big(T(e)x - eT(x)\Big)$$
$$- \binom{n-1}{n-2}\Big(T(x)e - xT(e)\Big) = 0.$$

Using relation (1.5) in above relation we get nT(x) = (n − 2)T(x), which implies using torsion restriction T(x) = 0 and gives D = G. This ascertainment enables us to combine the given two relations into only one relation

$$D(x^n) = D(x^{n-1})x + x^{n-1}D(x). \qquad (1.6)$$

Putting e for x in the above relation we obtain

D(e) = 0. (1.7)

Let y be any element of center Z(R). Putting x + y for x in the relation (1.6), we get

$$\sum_{i=0}^{n} \binom{n}{i} D(x^{n-i}y^i) = \Big(\sum_{i=0}^{n-1} \binom{n-1}{i} D(x^{n-1-i}y^i)\Big)(x+y)$$
$$+ \Big(\sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i\Big)D(x+y).$$

Using (1.6) and rearranging the preceding relation to include terms with an equal number of components of y, we get

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where fi(x, y) stands for the expression of terms involving i factors of y, that is

$$f_i(x,y) = \binom{n}{i} D(x^{n-i}y^i) - \binom{n-1}{i}\Big(D(x^{n-1-i}y^i)x + (x^{n-1-i}y^i)D(x)\Big) - \binom{n-1}{i-1}\Big(D(x^{n-i}y^i)y + (x^{n-i}y^i)D(y)\Big).$$

Replacing x by x + 2y, x + 3y,..., x + (n − 1)y in turn in the relation (1.6) and expressing the resulting system of (n − 1) homogeneous equations of the variables fi(x, y), i = 1, 2, . . . , n − 1, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

The determinant of this matrix is not zero, indicating that the system has a simple solution. In example, substituting the identity element e for y, we get

$$f_{n-2}(x,e) = \binom{n}{n-2} D(x^2) - \binom{n-1}{n-2}\Big(D(x)x + xD(x)\Big) - \binom{n-1}{n-3}\Big(D(x^2)e + x^2D(e)\Big) = 0.$$

Using (1.7) in the above relation it reduces to

$$\frac{n(n-1)}{2}D(x^2) = (n-1)(D(x)x + xD(x)) + \frac{(n-1)(n-2)}{2}D(x^2),$$

Since R is n!-torsion free, the above relation reduces to

$D(x^2) = D(x)x + xD(x)$

For every x in R. In other words, D is a Jordan derivative of R. Cusack's application of Herstein's theorem leads to the conclusion that D is a derivation, completing the proof. **Remark 1.2.** The foregoing theorem raises the obvious issue of whether the conclusion can be proven without the identity element. Unfortunately, we were unable to establish the preceding thesis generally. Is it possible to establish the following theorem without assuming a ring has an identity element?

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References

[1] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev: Rings with generalized identities, Marcel Dekker, Inc., New York, (1996).

[2] M. Brešar, J. Vukman: Jordan derivations on prime rings, Bull. Austral. Math. Soc. 37, 321–322 (1988).

[3] M. Brešar: Jordan derivations on semiprime rings, Proc. Amer. Math. Soc. 104, 1003–1006 (1988).

[4] M. Brešar: Jordan mappings of semiprime rings, J. Algebra 127, 218–228 (1989)

[5] J. Cusack: Jordan derivations on rings, Proc. Amer. Math. Soc. 53, 321–324 (1975). [6] I. N. Herstein: Jordan derivations of prime rings, Proc. Amer. Math. Soc. 8, 1104–1119 (1957).

[7] N. Rehman, N. Širovnik, T. Bano: A result on functional equations related to derivations and standard operator algebras, preprint.

[8] N. Širovnik: On functional equations related to derivations in semiprime rings and standard operator algebras, Glasnik Mat. 47, 95–104 (2012).

[9] J. Vukman, I. Kosi-Ulbl, D. Eremita: On certain equations in rings, Bull. Austral. Math. Soc. 71, 53–60 (2005).

[10] J. Vukman: Some remarks on derivations in semiprime rings and standard operator algebras, Glasnik. Mat. Vol. 46, 43–48(2011).