Analysing Numerical Roots of Quadratic Equations and its Importance

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Abstract

This research paper introduces an alternative method to derive the general formula for solving quadratic equations, suitable for instructing both high school and university students. Additionally, it explores the application of power series solutions when dealing with equations containing very small values. The paper also provides an examination of significant digit considerations in the context of quadratic equation roots.

Keywords— Quadratic equation; Exact roots; School mathematical speech; Significant digits.

I. Introduction

In today's educational landscape, pedagogy plays a pivotal role in the teaching and learning processes across various educational levels. A plethora of diverse learning strategies, such as the flipped classroom, project-based learning, meaningful learning, and the grid technique, have been employed to enhance the educational experience [1-3]. In the realm of textbooks, there's a recurring trend wherein updates to content are accompanied by improvements in pedagogical approaches. New editions of textbooks (e.g., [4-18]) typically introduce favorable changes, including more illustrative examples, increased exercise sets, the use of multiple ink colors, and enhanced visual aids. Some editions also incorporate the integration of mathematical software tools like Maple, Matlab, GeoGebra, and Excel [19-24]. However, amidst these updates, certain content elements remain unchanged, perpetuating what is referred to as the "school mathematical discourse" (dME).

In the domain of mathematics education, the school mathematical discourse (dME) encompasses all the language and reasoning introduced in the classroom. It is a system that, at times, imposes standardized arguments, meanings, and procedures, potentially leading to symbolic violence [25-26]. This discourse remains static even when innovations in teaching methods are introduced, as the core concepts taught remain unaltered [27]. Often, teachers are solely held accountable for learning difficulties in mathematics, overlooking the influence of textbooks, which are also shaped by the prevailing school mathematical discourse [27].

This standardized discourse is reflected in textbooks across various educational and professional fields. For instance, in control engineering texts [10], the method of determining the transfer function remains unchanged between updated editions. Similar instances occur in other disciplines [11]. In the realm of business administration, fundamental concepts like financial ratios (e.g., Net Present Value, Internal Rate of Return, Balance Point) are consistently presented across various textbooks [28]. However, the analysis of these coefficients should ideally be comprehensive and integrated to provide well-rounded, timely, and objective insights for investment decisions. While many textbooks adhere to the dME in presenting these concepts, some works in various fields have attempted to transcend it.

For example, in [29], an algebraic solution was proposed for a bias circuit with two rectifier diodes in series, offering an alternative to the graphical method typically used in teaching analog electronics. Similarly, in [32], approximations were introduced for the error function and normal cumulative function, aiming to revolutionize the teaching of probability calculations involving the normal distribution.

This paper presents an alternative approach to derive the general formula for solving second-degree algebraic equations, suitable for high school and engineering students. It also explores a solution method using power series, as introduced in [33], catering to students without an extensive mathematical background and aiming to combat the dME. The objective is to foster a renewed interest in mathematics among students and educators in the field, and the paper is structured as follows: Section 2 highlights the significance of the quadratic equation. Section 3 presents the alternative derivation of the general quadratic equation solution. Section 4 delves into the power series solution. Section 5 features two case studies: one on computational implications when rationalizing the general formula and another comparing the general formula's solution with the power series solution. Finally, Section 6 presents the conclusions drawn from this research.

The Significance of the Quadratic Equation

The quadratic equation, expressed as:

$$az^2 + bz + c = 0, \quad (1)$$

represents a parabolic curve and holds immense importance in the fields of physics and engineering. In physics, it plays a pivotal role in solving mathematical problems that ultimately lead to its solution [34]. Moreover, in the realm of ordinary differential equations, its utility is indispensable. Solving linear equations often necessitates finding the roots of the characteristic equation, which frequently takes the form of a second-degree algebraic equation [6,7].

In various engineering disciplines, particularly electronics, the quadratic equation is crucial. For instance, in the analysis of field effect transistor (FET) polarization, one encounters Shockley's second-degree equation, which requires solving [35,36]. Similarly, in electric circuit theory, solving quadratic equations is fundamental. In certain analyses, such as determining frequency responses using Bode diagrams [10], or assessing damping criteria in the response of analog filters [37,38], solving quadratic equations becomes essential.

It's worth noting that the study of quadratic equations begins in introductory algebra courses in secondary education in Mexico and is further explored in high school mathematics courses. However, numerous educators and researchers have proposed innovative approaches to teaching quadratic equations and understanding their geometric properties. For instance, in [39], a didactic design was introduced to reinterpret the concept of a parabola and its graphical representations in the context of real-world scenarios, like a person's motion.

In traditional algebra textbooks [4], quadratic equation solution methods are typically presented using classical algebraic techniques. Additionally, students learn about notable products, such as perfect square trinomials or difference of squares, when the equation exhibits these characteristics. However, the general formula for solving quadratic equations remains invaluable when coefficients make it challenging to find immediate solutions.

In [4], a proof is provided by adding terms and completing squares to derive the general formula for solving quadratic equations, expressed as:

$$z = (-b \pm \sqrt{(b^2 - 4ac)}) / (2a) \quad (2)$$

III. OBTAINING IN AN ALTERNATIVE WAY THE GENERAL FORMULA

We begin by making $z = x + yi$, and substituting in equation (1), where z is a complex number with real part givenby x, and imaginary part given by y, simplifying we obtain

$$ax^2 + 2iaxy - ay^2 + bx + iby + c = 0. \quad (3)$$

From (3) we get the real part and the imaginary part.

Solving for the imaginary part we have

$$x = -\frac{b}{2a} \quad (4)$$

Substituting (4) in the real part we obtain

$$-\frac{h^2}{2a}\, ay^2 + c = 0. \quad (5)$$

From (5) we can see that it is easy to solve for y, solving

$$y = \pm\sqrt{\frac{4ca - b^2}{2a}}. \quad (6)$$

Substituting (4) and (6) into $z = x + yi$, we have

$$z = -\frac{b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a} i. \quad (7)$$

Furthermore, taking into account that $i = \sqrt{-1}$, equation

(7) can be simplified, obtaining then the general formula givenby (2). However, when $b^2 \gg 4ac$, the numerical difference inthe numerator can be very small. In these cases, it is useful to make use of double precision variables [40] when formula (2) is implemented in a programming language. To minimize the effects of cancellation due to subtraction, equation (7) can be rewritten as

$$z = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}. \quad (8)$$

I. SOLUTION BY POWER SERIES

The solution through power series is possible when there are perturbative parameters of ε, being a constant whose restriction is $\varepsilon \ll 1$. Consider the quadratic equation given by with solutions $x_1 = -5$ and $x_2 = 0$ when $c = 0.14$)

$$x^2 + k_1 \varepsilon x - k_2 = 0, \quad (9)$$

To see the behavior of the significant digits of the roots of this equation, it is considered to give different values to c, such that $c \ll 1$. Then, to determine the roots of (14) the formulas of (2) and (8) will be used. In addition, we will

where k_1 is a real integer constant and k_2 a real constant. To solve this equation, perturbative methods used [33] are used. Now consider

$$x \approx x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \quad (10)$$

using the first three terms of the series and substituting in (9) we have

$$\varepsilon^4 {x_2}^2 + \varepsilon^3 (k_1 x_2 + 2 x_1 x_2) + \varepsilon^2 (k_1 x_1 + 2 x_0 x_2 + {x_1}^2) + \varepsilon (k_1 x_0 + 2 x_0 x_1 + {x_0}^2) = k_2, \quad (11)$$

equating the coefficients of ε^n, we obtain, the system given by

$$\begin{aligned} \varepsilon^0&: \quad {x_0}^2 = k_2, \\ \varepsilon^1&: \quad k_1 x_0 + 2 x_0 x_1 = 0, \\ \varepsilon^2&: \quad k_1 x_1 + 2 x_0 x_2 + {x_1}^2 = 0. \end{aligned} \quad (12)$$

Solving the system of equations given by (12), and substituting the solutions of x_0, x_1 y x_2 in (10), the solutions in power series are given by

$$\begin{aligned} x_{s1} &= \sqrt{k_2} - \frac{k_1 \varepsilon}{2} + \frac{\sqrt{k_2}{k_1}^2 \varepsilon^2}{8}, \\ x_{s2} &= -\sqrt{k_2} - \frac{k_1 \varepsilon}{2} - \frac{\sqrt{k_2}{k_1}^2 \varepsilon^2}{8}. \end{aligned} \quad (13)$$

V. STUDY CASES

In the first case study, we analyze the behavior of the significant digits for the when solving the quadratic equation using (2) against (8). The second case study discusses the comparison of the solution calculated by the general formula against the solution obtained with perturbation in power series.For the development of the case studies presented in this work, significant digits remains constant, whereas this does not occur for eq. (8). In the case of x_2 in eq. (8), the significant digits remain constant, but not for eq. (2).

In order to show the accuracy of the roots of the equations under discussion, we will use the formula (15) to obtain the significant digits [42, 43], given by the mathematical software Maple 15 and its fsolve command have been used, which is an inter-build function that has a Newton-Raphson (NR) algorithm and that we will use as an exact solution to validate the results presented in this article.

A. Significant digits using the general formula and itsrationalization

We will consider the quadratic equation given by

$$x^2 + 5x - c = 0,$$

with solutions $x_1 = -5$ and $x_2 = 0$ when $c = 0$.

To see the behavior of the significant digits of the roots of this equation, it is considered to give different values to c, such that $c \ll 1$. Then, to determine the roots of (14) the formulas of (2) and (8) will be used. In addition, we will consider that the exact solutions are those that have been determined with the numerical differentiation NR. In [30, 31] algorithms and their implementation are presented using a programming language such as C [40] or Fortran [41]. Table 1 presents the comparison of using equations (2) and (8)against NR for $1E-5 \leq c \leq 1$.

TABLE I. COMPARISON OF SIGNIFICANT DIGITS BETWEEN EQUATIONS (2) AND (8) AGAINST NR TABLE STYLES

c values	Roots with (2) x_1 x_2	Roots with (8) x_1 x_2	Roots using NR. x_1 x_2
1	-5.192582405 0.1925824035	-5.192582406 0.1925824035	-5.192582404 0.1925824036
1E-1	-5.019920635 0.01992063350	-5.019920676 0.01992063367	-5.019920634 0.01992063367
1E-2	-5.001999200 0.00199920064	-5.001999550 0.00199920064	-5.001999201 0.01999200639
1E-3	-5.000199990 0.00019999200	-5.000200008 0.0001999920	-5.000199992 0.00019999201
1E-4	-5.000020000 0.00002000000	-5.000000000 0.0000199999	-5.000020000 0.00001999992
1E-5	-5.000002000 0.00000200000	-5.000000000 0.00000199999	-5.000002000 0.00000199999

The formulas for solving the quadratic equation (8) and (2) are algebraically the same expression. In other words, to obtain eq. (8), it has proceeded to rationalize the numerator in (2), however, carrying out this process has a very important implication in numerical calculations, that the traditional books, following the dME, do not warn the student. In Table 1 we can observe the behavior of the significant digits of both

roots in equations (2) and (8) as the value of c is changed. Forexample, if $c = 1E - 5$, for x_1 in eq. (2) the number of significant digits remains constant, whereas this does not occur for eq. (8). In the case of x_2 in eq. (8), the significant digits remain constant, but not for eq. (2).

In order to show the accuracy of the roots of the equations under discussion, we will use the formula (15) to obtain the significant digits [42, 43], given by

$$\mathrm{SD} = -\log_{10}\left|\frac{x - \tilde{x}}{x}\right|, \qquad (15)$$

where SD is the number significant digits, x is the exact value computed with NR, $\tilde{x}$ are the roots calculated with equations (2) and (8)

Figure 1 shows the graphs of significant digits of the data is shown in table 1. In eq. (2) the number of significant digits for x_1 remains constant, in this case 11, while for x_2 they decrease as c tends to 1, that is, 5 significant digits. On the other hand, in the case of eq. (8) x_2 is the root that keeps the significant digits constant, while x_1 shows a reduction of these. From this figure an inverse behavior can be seen in the roots of equations (2) and (8).

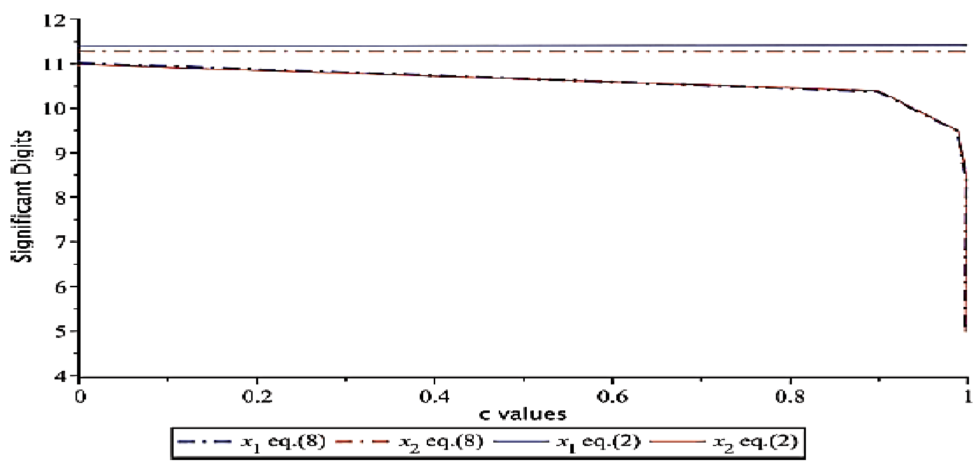

Fig. 1. Comparison of significant digits in the roots from equations (2) and (8).

From table 1 and Figure 1 it can be concluded that the advantage of using the rationalized formula (8) when we havethe condition $b^2 \gg 4ac$, is that the number of significant digits can be maintained with good precision. To do this, in the denominator, both b and $\sqrt{b^2 - 4ac}$ must have the same sign. When this condition is not met, it is better to use formula (2).

B. Perturbative solution with power series.

Consider the constants $k_1 = 5$, $k_2 = 1$, and $\varepsilon = 0.00$,substituting into (9) we have

$$x^2 + 0.005x - 1 = 0. \quad (14)$$

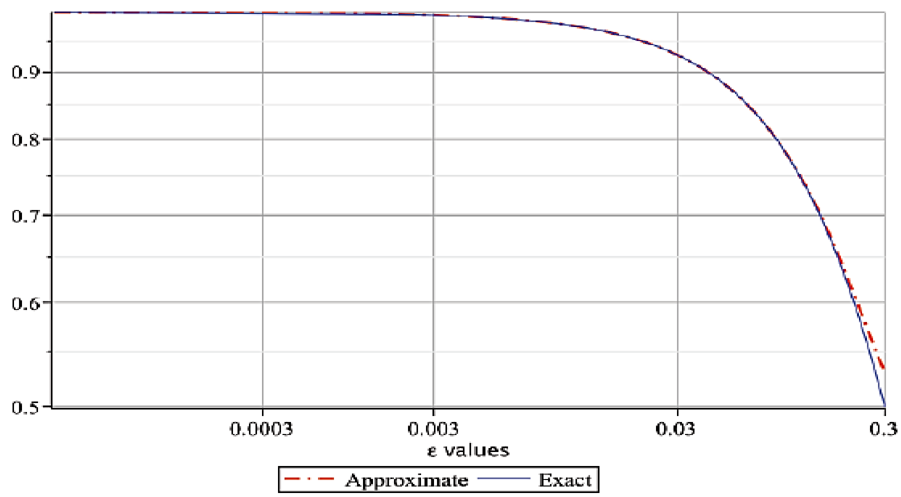

Fig. 2. Comparison between the exact positive root and x_{s1} with three terms in series expansion. For small values of ε the approximate solution is closer to exact solution.

VI. CONCLUSIONS

In this article an alternative deduction has been presented using a complex expression for the general formula that allows solving quadratic equations. In the same way, the perturbative solution was presented through power series.

Two case studies were also presented. In the first, the behavior of the significant digits of the roots of a quadratic equation was discussed. This behavior, following the dME in traditional algebra books is not analyzed, nor is anything commented on about it. On the other hand, in [30] the effect of rationalization is commented, however, also following the dME, the reader is presented with a detailed analysis of the advantages and disadvantages of rationalizing the general formula for solving these equations.

In the second case study, the solution was discussed through power series, which was obtained through perturbation methods when there is a small parameter such that $\varepsilon \gg 1$. In the example presented, 10 significant digits were determined, which shows a high accuracy when using this method considering the first three terms of the expansion to approximate the root. Likewise, Figure 1 illustrated that the accuracy decreases if ε we make it grow. Figure 1 suggests that to obtain an accuracy of approximately 95%, it is necessary touse a value $\varepsilon < 0.01$.

In this article we have presented some alternatives that can be used in algebra courses in teaching the quadratic equation. We believe that this work will provide an enriching experience, both for students and teachers in the teaching-learning processes, with which the school mathematical discourse that has been present in traditional books and in classrooms can be overcome from another perspective. This will allow proposingother teaching strategies for the benefit of students within the discipline of mathematics.

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